Illuminated by dialectical and historical materialism

MATHEMATICS:

Its Essential Nature and Objective Laws of Development

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Translation editing by Esther R. Phillips from translations by Gary Laison and by Michael Meo with Charles Aronson [from the 1956 Russian edition].

Editorial note: We present two philosophical essays that were omitted from the 1963 translation—by the American Mathematical Society—of the much-respected Soviet exposition, *Mathematics: Its Content, Methods and Meaning,* edited by A.D. Aleksandrov, A.N. Kolmogorov, and M.A. Lavrent'ev. These essays were the concluding sections of Chapter I, "A General View of Mathematics," written by Aleksandrov with assistance from V.A. Zalgaller. For those who have not read the chapter, we preface the essays with a summary of the portion previously published in English. A comment on the censorship aspect is appended.

Summary of Sections 1 through 7, prepared by Irving Adler

Aleksandrov begins by listing some characteristic features of mathematics: "its abstractness, its precision, its logical rigor, the indisputable character of its conclusions, and finally, the exceptionally broad range of its applications." In a preliminary clarification of these characteristic features, with emphasis on specific examples from the history of arithmetic, algebra and analysis, some of the points he makes are:

– All the abstract concepts of mathematics are "connected with actual life both in their origin and in their applications."

– Theorems in mathematics must be proved by logical argument from axioms.
– "The rigor of mathematics is not absolute. It is in a process of continual development."
– "Mathematical concepts. . . are brought into being by a series of successive abstractions and generalizations, each resting on a combination of experience with preceding abstract concepts."
– ". . . The development of mathematics is a process of conflict among the many contrasting elements: the concrete and the abstract, the particular and the general, the formal and the material, the finite and the infinite, the discrete and the continuous, and so forth."
– "The old theories, by giving rise to new and profound problems, outgrow themselves, as it were, and demand for further progress new forms and new ideas."

As a result, the growth of mathematics has led to a succession of qualitative changes. Aleksandrov discerns four distinct stages in the development of methematics:

1. The period of the formation of arithmetic and geometry as collections of rules deduced from experience and immediately connected with practical life.
2. The period of elementary mathematics, dealing with constant magnitudes.
3. The period of the birth and development of analysis, the mathematics of motion and change, which embraces the study of variable magnitudes.
4. The period of contemporary mathematics, characterized by an immense extension of the subject matter of mathematics and its applications; the formation of general concepts on a new and higher level of abstraction; the dominance of the set-theoretic point of view; and the interpenetration of all of the various branches of mathematics. *"Contemporary mathematics is the mathematics of all possible (in general, variable) quantitative relations and interdependences among magnitudes."*

His summary and conclusions are then given in Sections 8 and 9, which follow:

SECTION 8

The Essential Nature of Mathematics

1. Based on what has been discussed already, we may now turn to some general conclusions concerning the nature of mathematics.

The nature of mathematics was described by Engels in a section of *Anti-Duhring,* and we quote this remarkable passage here. The reader will easily recognize in Engels' formulation what we have already said, for example, with regard to arithmetic and geometry—and understandably so— since we explained the actual history of the origin and development of mathematics, guided by an understanding of dialectical materialism. Dialectical materialism leads to true results precisely because it does not superficially impose anything on reality, but examines the facts as they are, i.e., in their necessary relationships and development.

Engels begins his discussion of the nature of mathematics with some critical remarks about the absurd opinions of Duhring, in particular the false opinion that mathematics is engaged in the creation of "pure reason", independent of experience. Engels wrote:

> But it is not at all true that in pure mathematics the mind deals only with its own creations and imaginations. The concepts of number and form have not been derived from any source other than the world of reality. The ten fingers on which men learnt to count, that is, to carry out the first arithmetical operation, may be anything else, but they are certainly not a free creation of the mind. Counting requires not only objects that can be counted, but also the ability to exclude all properties of the objects considered other than their number—and this ability is the product of a long historical evolution based on experience. Like the idea of number, so the idea of form is derived exclusively from the external world, and does not arise in the mind as a product of pure thought. There must be things which have shape and whose shapes are compared before anyone can arrive at the idea of form. Pure mathematics deal with the space forms and quantity relations of the real world—that is, with material which is very real indeed. The fact that this material appears in an extremely abstract form can only superficially conceal its origin in the external world. But in order to make it possible to investigate these forms and relations in their pure state, it is necessary to abstract them entirely from their content, to put the content aside as irrelevant; hence we get the point without dimensions, lines without breadth and thickness, *a* and *b* and *x* and *y*, constants and variables; and only at the very end of all these do we reach for the first time the free creations and imaginations of the mind, that is to say, imaginary magnitudes. Even the apparent derivation of mathematical magnitudes from each other does not prove their *a priori* origin, but only their rational interconnection. Before it was possible to arrive at the idea of deducing the *form* of a cylinder from the rotation of a rectangle about one of its sides, a number of real rectangles and cylinders, in however imperfect a form, must have been examined. Like all other sciences, mathematics arose out of the *needs* of men; from the measurement of land and of the content of vessels, from the computation of time and mechanics. But, as in every department of thought, at a certain stage of development the laws abstracted from the real world become divorced from the real world, and are set over against it as something independent, as laws coming from outside, to which the world has to conform. This took place in society and in the state, and in this way, and not otherwise, *pure* mathematics is subsequently *applied* to the world, although it is borrowed from this same world and only represents one section of its forms of interconnection—and it is only just precisely because of this that it can be applied at all. [*Anti-Duhring*, New York 1939, pp. 45-46.]

2. Thus, Engels emphasizes that mathematics reflects reality, that it arose from practical needs of people, and that its first concepts and principles came as a result of a long historical development grounded in experience. We have already examined this in abundant detail in the examples of arithmetic and geometry. We have convinced ourselves, in particular, that the ideas of number or magnitude and of geometrical figures arose in this way, and that they reflect the real quantitative relations and spatial forms of reality. The fundamental ideas of analysis reflect real quantitative relations in exactly the same way. They are built up gradually, beginning with generalizations of enormous amounts of concrete material; thus, the concept of function is a reflection, in generalized abstract form, of various relations between real quantities.

Summarizing all this, Engels arrives at the fundamental conclusion: *mathematics has real matter as its subject, but considers it in complete abstraction from its concrete contents and qualitative peculiarities.* In this respect it is clear that mathematics must be distinguished from the natural sciences, and Engels clearly makes this distinction [*Anti-Duhring*, pp 45-47].

The possibility of abstractly examining the subject of mathematics is objectively based in the subject itself. Its general forms, relations, interconnections and laws—independent of the specific peculiarities or concrete content— exist objectively, independent of our knowledge of them. Thus, the existence of number as an objective property of sets of objects, the independence of numerical relationships from the specific properties of the objects, and the richness of these relationships, made arithmetic possible. Where such common forms and relations, independent of content, do *not* exist, there mathematical examination is impossible.

3. The aforementioned fundamental characteristic of mathematics determines other characteristic properties. In Section 2 we examined some of these special features in the case of arithmetic. These are: the specific "formal language", the wideness of application, the abstraction of results from experience, their logical inevitability, and their persuasiveness. The theoretical character of mathematics is clearly an essential feature of it, and we now examine this feature in detail.

If we abstract, for example, the idea of number from its concrete base and consider pure numbers in general, apart from any relation to one or another concrete collection of objects, then it goes without saying that we are not able to carry out experiments on such abstract numbers. Remaining at this level of abstraction without returning to the concrete object, it is possible to get results about numbers only by means of arguments based on the *concept* of number itself. The same applies, of course, to all other mathematical results. Remaining within the limits of pure geometry, i.e., considering geometrical figures completely abstracted from any qualitative, concrete content, we can derive new results only by reasoning from the very *concept* of this or that figure, from the basic concepts of geometry or from the axioms themselves. Thus, properties of a circle are deduced from the idea of a circle as the geometric locus of points equidistant from a given point, and by no means is each theorem verified by experience.

Therefore, *the abstract character of mathematics is already predetermined by the fact that mathematical theorems are proved only by reasoning, based on the concepts themselves.*

It is possible to say that in mathematics we investigate quantitative relations, keeping in mind only what is contained in the definitions themselves. Correspondingly, mathematical results are obtained by arguments derived from the definitions. Of course, it would be incorrect to interpret this too literally and to suppose that sufficiently rigorous definitions of mathematical ideas were actually formulated before the creation of the corresponding mathematical theories; indeed, the concepts themselves were made more accurate in the course of the development of the theory and as a result of this development. A profound analysis of the idea of whole number, as well as a precise formulation of the axioms of geometry was not carried out in antiquity but at the end of the 19th century. It would be even more wrong to think that there is some kind of class of absolutely, precisely determined mathematical ideas. Every concept, however precisely defined it may seem, is nevertheless mutable—it evolves and is made more precise with the development of the science. This is completely demonstrated by the development of mathematics in relation to all its concepts, and it only confirms once again the fundamental proposition of dialectics that there is nothing in the world which is immutable and not subject to development. Thus, with respect to mathematical ideas, we may speak, in the first place, only of sufficient, but not of absolute, precision, and, in the second place, we must keep in mind that the precision and clarity of its definitions and the depth of its analysis evolved with the development of mathematics. On the subject of the changing character of mathematical concepts we shall have more to say in the following section; but now, keeping in mind the above remarks, we consider in detail the adequacy of the precision.

This precision of the mathematical concepts—along with the general applicability of logic itself—appears to be the reason for the inner persuasiveness and logical necessity which are characteristic of mathematical results. The inevitability of the theoretical results of mathematics gives rise to the erroneous idea that mathematics has its foundation in pure thought, that it is *a priori* and not derived from experience, that it does not reflect reality. The famous German philosopher Kant, for example, arrived at this point of view. This deeply erroneous ideological notion arises, in particular, when mathematics is considered in its finished form and not in terms of its actual origins and development. But this approach is quite sterile, for the simple reason that it does not correspond to the actual state of things. For it is firmly established that mathematics is not *a priori*, but arose from experience. In fact, the actual origins of geometry were written about by Eudemus of Rhodes, whom we quoted in Section 3.

Not only the concepts of mathematics, but also its results and its methods reflect reality. This important point is stated clearly by Engels, who writes: "Even the apparent derivation of mathematical magnitudes from each other does not prove their *a priori* origin, but only their rational inter-connection." Mathematical results and proofs arose as reflections of real relations which people investigated in their experience. The addition of numbers reflects the actual combination of several objects aggregated into one. The well-known proofs of theorems about equality of triangles, in which one speaks of their superposition, certainly have their origin in the operation of actually applying one object to another; this constantly takes place in the comparison of their sizes. The calculation of volumes by integration reflects in abstract form the real possibility of building up bodies from fine layers, or of slicing them into such layers. More complicated mathematical proofs are results of a further development originating from this material foundation.

4. The complete abstraction of the objects of mathematics from everything concrete, and the theoretical character of the mathematical results which are based on it, have as a consequence another important feature of mathematics: in mathematics we investigate not only quantitative relations and spatial forms which are immediately abstracted from reality but also relations and forms which are defined within mathematics on the basis of concepts and theories which have already been put together. It is just this feature of mathematics which Engels considers when, referring to the origin of the concepts of points, lines, constant and variable quantities, he says: "Only at the very end of all these do we reach for the very first time the free creations and imaginations of the mind, that is to say, imaginary magnitudes."

The historical fact is that imaginary numbers were not taken from reality in the same sense as, say, integers. They appeared originally within mathematics itself, a product of the necessary development of algebra, as roots of equations of the form $x^2 = -a$ (where $a > 0$). Although, gradually, operations with them were carried out quite freely, their real meaning remained for a long time unclear, which is why they acquired the name "imaginary". Subsequently their geometric interpretation was discovered, and numerous important applications were found. In precisely the same way, Lobachevskian geometry originated as the creative product of the great scientist; he did not see its real significance and consequently named it "imaginary geometry". However, it was not free play of mind but the inevitable result of the fundamental concepts of geometry, and Lobachevsky considered it as a possible theory of spatial forms and relations. Thus, it is not possible to interpret "the free creations and imaginations" of which Engels speaks as simple arbitrariness of thought. Free creation in science: this is a realization of logical necessity, determined by the concepts and the initial positions taken from experience.

In the most recent stage of the development of mathematics, the beginning of which can be precisely placed at the time of the construction of Lobachevsky's geometry and the precise meaning of imaginary numbers, new concepts and theories appeared and continue to appear; these are based on previously constructed concepts and theories which need not borrow directly from reality. Mathematics defines and investigates the possible forms of reality; this is one of the decisive characteristics of the recent stage of its development.

A correct understanding of this characteristic is provided by the theory of knowledge of dialectical materialism. Lenin wrote: "Knowledge is the reflection of nature by man. But this is not a simple, not an immediate, not a complete reflection, but the process of a series of abstractions, the formation and development of concepts, laws, etc..." [*Philosophical Notebooks,* Moscow 1963 p. 182]. Metaphysical materialism also recognizes that knowledge, in particular mathematical knowledge, is a reflection of nature. However, as Lenin notes, the weakness of metaphysical materialism is its inability to apply dialectics to the theory of reflection [ibid, p. 362]. Metaphysical materialism does not understand the complexity of this reflection, does not understand that it goes through a series of abstractions by the formation of new concepts, by the construction of new theories on the basis of concepts and theories previously constructed, and by the examination not only of the data of experience but of its possibilities. This transition from data to possibilities is already manifested in the formation of such concepts as arbitrary whole number or infinite straight line, since there is no data in experience of either arbitrarily large integers or infinite extension. But when the concept of number is crystallized, the possibility of the infinite continuation of the number sequence is manifested from the concept itself and from the law of formation of successive numbers by the addition of a unit. In the same way, the extension of a line segment reveals the possibility of its infinite extension, expressed in Euclid's second postulate: "Every straight line can be extended infinitely". The subsequent process of abstraction led to the concepts of the entire sequence of natural numbers and *all* of the infinite straight lines. In the most recent stage of the development of mathematics the construction of theories has been qualitatively new, passing through a sequence of abstractions and formations of concepts. But, going back along the path of these abstractions, we see that mathematics is by no means separated from reality. What is new arises on the basis of the reflection of reality, as a result of the logic of the subject itself, and particularly by means of the return to reality in applications to problems of physics and technology. So it was with imaginary numbers. It is also true in relation to other mathematical theories, however abstract they may be. A characteristic example is provided by the theory of spaces of n-dimensions. Such spaces were invented as generalizations of Euclidean geometry in conjunction with the development of algebra and analysis, under the influence of mechanics and physics. The combination of these ideas led Riemann to the construction of the general theory which, developed further by other mathematicians, found a series of important application and, in the end, provided a ready mathematical apparatus for Einstein's construction of the general theory of relativity (more precisely, the theory of gravitation). It is no accident that abstract geometric theories found such brilliant applications, nor was it a result of "preordained harmony of nature and reason"; rather, it was a result of the fact that these theories grew out of geometry, which was directly grounded in experience, and that they were related, by their creators, to problems of investigating real space. Riemann, in particular, clearly foresaw the connection of his theory with the theory of gravitation.

Thus, in the development of mathematics, there is the law of the motion of knowledge formulated by V. I. Lenin: "Thought proceeding from the concrete to the abstract--provided it is *correct. . .* does not get away *from* the truth but comes closer to it. The abstraction of *matter,* of a *law* of nature, the abstraction of *value,* etc., in short *all* scientific (correct, serious, not absurd) abstractions reflect nature more deeply, truly and *completely.* From living perception to abstract thought, *and from this to practice*--such is the dialectical path of the cognition of *truth,* of the cognition of objective "reality". [ibid, p 171.]

From what has been said it is clear that the idealist view—that mathematical theories constitute merely conventional schemes chosen to describe the data of experience, or to "order the stream of sensations" on the basis of the "principle of economy of thought"--is completely false.

Engels notes (as quoted earlier) that the propositions of mathematics, abstracted from the real world as if they were opposed to it, are applied to its study as some ready-made schema. For example, we continually make use of computations in the form of finished (tabulated) numbers. This is even more true of the theories arising at higher stages of abstraction. In the example already discussed, Riemannian geometry served as a readily available mathematical schema for the theory of gravitation. But Engels explains that the possibility of such an application of mathematics to the investigation of the real world is based on the fact that mathematics was borrowed from this world, and only expresses a part of its inherent forms of relations—indeed, *only because of this* can it be applied at all. The fact that many theories are created within mathematics itself does not change any of this. The development of applications of formal theories to reality is absolutely not a matter of convention; this development occurs as a consequence of the logic of the subject itself. In any case, mathematical theories reflect reality—the only difference among them being that the reflection is more immediate in some cases, while in others it goes through a series of abstractions, conceptualizations, etc.

5. The most recent stage in the development of mathematics is characterized not only by higher levels of abstraction; it is further characterized by the essential widening of its subject matter, by going beyond the limits of the initial concepts of quantitative relations and spatial forms.

Figures in a space of several dimensions—or of infinite dimensions—are not, of course, spatial forms in the usual sense in which we understand them when we have in mind ordinary real space, rather than the abstract spaces of mathematics. Such spaces have real meaning and reflect in an abstract way definite forms of analogous reality; for this reason, in contrast to ordinary real space, we might call them "space-like". In speaking of space of several dimensions, or of figures in it, we attach new content to the concept of space, so that it is necessary to distinguish clearly between, on the one hand, the generalized, abstract concept of space in mathematics and, on the other, the concept of space in its original sense as the universal form of the existence of matter.

The emergence at the end of the last century of the new discipline of mathematical logic, since developed extensively, will serve as another example of the way the subject matter of mathematics has broken free of the limitation to spatial forms and quantitative relations, in the original meaning of these terms. The object of consideration in this discipline is the structure of mathematical proofs; that is, it studies which propositions may be derived from given premises by prescribed rules. It investigates this subject, as is characteristic of mathematics, in complete abstraction from the content, thus replacing propositions by formulas and rules of inference by the principles of operating with these formulas. Relations between premises and conclusions, axioms and theorems, of course, do not reduce to spatial forms or to quantitative relations in their usual sense of relations between numerical values.

As another example, we consider the theory of groups which may be understood as the study of symmetries in the most general sense. The change in the symmetries of a crystal, say, in sulfur passing from rhomboidal to prismatic form, is a fundamental qualitative change of the state of the substance. In this sense, group theory is the study of quantities or defined qualities of an object, changes in which are accompanied by fundamental changes in the object itself.

A consequence of the extension of the subject matter of mathematics is the substantial extension of our understanding of quantitative relations and spatial forms. What then are the characteristic general features of this expansion in the subject matter of mathematics?

If we answer this question not by enumerating but by attempting to elucidate the common features of these subjects in all their various forms, then the answer is found essentially in Engels. It suffices to draw attention to his treatment not only of the subject matter of mathematics but also of the way in which mathematics deals with its subject matter: the complete abstraction of form and relations from their content. This abstract character of mathematics at the same time

provides us with a definition of its content.

The subject matter of mathematics consists of those forms and relations of reality which objectively have such a high degree of indifference towards content that they can be completely abstracted from this content and defined in a general way with such clarity and precision, preserving such a wealth of relations, that they provide a basis for the purely logical development of the theory. If we call these forms and relations quantitative in the general sense of the word, it is possible to say briefly that the subject of mathematics consists of quantitative relations and forms viewed purely abstractly.

Abstraction is by no means the privilege of mathematics alone. Other sciences, however, are primarily interested in the degree of conformity of their systems of abstraction to a clearly defined collection of data; one of their important problems is the task of investigating the limits of the applicability of the theoretical system to the collection of data and determining appropriate changes in the abstract system. Mathematics, on the contrary, while investigating general properties in full abstraction from specific data, examines these systems of abstraction themselves in their abstracted generality, outside the boundaries of their applicability to individual concrete phenomena. One can say that for mathematics the absoluteness of abstraction is characteristic.

It is just the indicated indifference to the content of the forms investigated in mathematics which defines the fundamental properties of mathematics: its theoretical character, the logical necessity and apparent immutability of its results, the origination from within of its new concepts and theories; just the indifference to content determines the special character of the applicability of mathematics. When we can translate a practical problem into the language of mathematics, we may, at the same time, "abstract ourselves" from the concrete second-stage characteristics of the problem, and, by making use of general formulae and theorems, obtain precise results. In this way the abstraction of mathematics constitutes its power; this abstraction is a practical necessity.

6. Returning now to Engels' opinions about mathematics we can see their depth and richness, and the possibility of developing them further. Not himself a mathematician, he was able to make such a profound analysis of this science not only because he was a thinker of genius, but mainly because he was able to use dialectical materialism, and was guided by it in his explanation of the essence of mathematics. It is therefore not strange that no one before Engels was able to give so profound and correct a solution to this problem. Great mathematicians were unable to resolve the problem in this manner.

It was exactly in this way that Lenin later gave an analysis of the problem of physics that surpassed anything done in this area.

This demonstrates yet again the knowledge and power provided by dialectical materialism; it demonstrates that it is not enough to possess knowledge of individual propositions; nor is it sufficient to be a creative scientific worker—it is also necessary to possess the correct general method, to master dialectical materialism. Without this the results of science either will seem a shapeless heap or will present themselves in a distorted way; instead of a true understanding of science there will be a false metaphysical idealist representation of it. So, for example, many mathematicians who do not possess dialectical materialism are either completely disoriented in the general questions concerning their science or treat them in a completely inaccurate way.*

At the time when Engels wrote *Anti-Duhring,* i.e., in 1876-1877, non-Euclidean geometry and the geometry of space of several dimensions were just gaining acceptance among mathematicians, the theory of groups had just been formulated, the theory of sets had just appeared, and mathematical logic had only just been born. It is therefore obvious that Engels could not have given a detailed discussion of the characteristic properties of the latest stage in the development of mathematics; nevertheless, we can find in his opinions hints for understanding them.

SECTION 9

The Laws of Development of Mathematics

In conclusion, we shall attempt to describe briefly the general laws of the development of mathematics.

1. Mathematics is not the creation of any one historical epoch or of any one people; it is the product of a series of epochs and the work of many generations. As we saw, its first ideas and propositions arose in earliest antiquity and had already been put into a coherent system more than two thousand years ago. Despite all the transformations of mathematics, its ideas and results are preserved in the transition from one epoch to another, as, for example, the laws of arithmetic or the Pythagorean theorem.

New theories contain the ones which precede them—extending, sharpening, completing, and generalizing them.

At the same time, it is clear from the brief outline of the history of mathematics presented above that its development is not simply an accumulation of new theories but includes essential qualitative changes. Correspondingly, the development of mathematics can be separated into a sequence of historical periods with the transitions between them marked by fundamental changes in the subject matter or the structure of this science.

* It is interesting to observe, for example, that the two eminent American geometers Veblen and Whitehead attempt to define what geometry is in their book *Foundations of Differential Geometry* and conclude that it is impossible to give such a definition except perhaps the following: "geometry is whatever geometers say it is".

Mathematics includes in its province all new areas of quantitative relations of reality. At the same time, the most important objects of mathematics were and remain the spatial forms and quantitative relations in the simple, most direct meanings of these terms, and mathematical comprehension of new connections and relations inevitably arise on the basis of and in connection with previously constructed systems of quantitative and spatial scientific representations.

Finally, the accumulation of results within mathematics itself necessarily leads to the ascent to new levels of abstraction and new generalizations of concepts and thereby to a deepening of the analysis of the original concepts.

As a great and powerful oak thickens old branches with new layers, puts out new branches, extends upwards, and deepens its roots downwards, so mathematics in its development adds new material to its already existing areas, forms new directions of inquiry, ascends to new heights of abstraction, and deepens its own foundations.

2. Mathematics has as its subject the real forms and relations of reality, but, as Engels said, in order to study these forms and relations in pure form it is necessary to isolate them completely from their content, to put the latter aside as irrelevant. However, forms and relations do not exist apart from content; mathematical forms and relations cannot be absolutely indifferent to content. Consequently mathematics, by its very nature, aspiring to accomplish that separation, attempts the impossible. *This is the fundamental contradiction at the heart of mathematics.* It is the specific manifestation in mathematics of the general contradictions in knowledge. The reflection in thought of any phenomenon, any aspect, any amount of reality coarsens and simplifies it, wrenching it away from its general connections in nature. When people, studying the properties of space, ascertained that it was Euclidean, it was an exceptionally important act of cognition, although it contained an error: the real properties of space were taken simply, schematically, in abstraction from matter. But without this there would simply have been no geometry, and on the basis of this abstraction (by internal deduction, as well as by the confrontation of the mathematical results with new data of another science) new geometrical theories were produced and strengthened.

The constant resolution and re-establishment of such contradictions at new levels of knowledge ever more closely approximating reality constitutes the essence of the development of knowledge. This concept of development, of course, ascribes a positive content to knowledge, an element of absolute truth in it. Knowledge advances in an ascending line, and it is not rendered worthless by an admixture of error.

The fundamental contradiction, which we have indicated, leads to others. We saw this in the example of the opposition of the discrete and the continuous. (In nature there is not an absolute separation between them, and their separation in mathematics inevitably made necessary the creation of entirely new ideas profoundly reflecting reality, while at the same time overcoming internal imperfections in existing mathematical theories.) Exactly in this way the contradictions between finite and infinite, abstract and concrete, form and content, etc. appear in mathematics as manifestations of its *fundamental contradiction* defined above. But the decisive factor in its manifestations is that, in abstracting from the concrete and linking up its abstract ideas, mathematics separates itself from experience and practice; but at the same time it proves to be a science (i.e., has significant cognitive value) to the extent that it rests on practice, to the extent that it proves to be not pure but applied mathematics. Speaking for the moment in Hegelian language, *pure mathematics continually "negates" itself as pure mathematics; if it did not do so it could not have scientific significance, could not develop, could not surmount the difficulties which inevitably arise in it.*

In their formal aspect mathematical theories stand apart from their real contents as so many schema for obtaining concrete results. Mathematics emerges in this way as a method for formulating quantitative laws of the natural sciences, as an apparatus for making use of its theory, as a means for solving problems in the natural sciences and technology. The significance of pure mathematics in the present epoch resides mainly in the mathematical method. And, as every method exists and is developed not for its own sake but for its applications, in connection with the content to which it is applied, so mathematics cannot exist and develop without applications. Here again is revealed the unity in contradiction: the general method stands in opposition to the concrete problem lem as a means of its solution; but itself arises from the generalization of concrete material and itself exists, develops, and finds justification only in the solution of concrete problems.

3. Social practice plays a determining role in the development of mathematics in three respects: it poses new problems for mathematics, stimulates its development in particular directions, and provides criteria for the validity of its results.

This can be seen with extraordinary clarity in the example of the origins of analysis. In the first place, it was developments in mechanics and technology which brought forward the problem of studying the dependence of variable quantities in the most general form. Archimedes came right to the edge of the differential and integral calculus. but remained nonetheless in the framework of problems in statics, while in modern times it was precisely the investigation of motion that produced the concepts of variable and function and made necessary the formalization of analysis. Newton could not have developed mechanics without developing the corresponding mathematical methods.

Secondly, it was precisely the needs of social production which prompted the posing and the solving of all these problems. This stimulus was not yet present either in ancient or medieval society. Finally, it is quite characteristic of mathematical analysis, in its beginning, that it found proofs for its results primarily in its application. Only for this

reason could it be developed without rigorous definitions of its fundamental ideas (variable, function, limit) which were not given until later. The validity of analysis was established by its applications to mechanics, physics, and technology.

What we have said applies to all periods of the development of mathematics. Beginning with the 17th century, mechanics, theoretical physics, and the problems of the new technology exerted an especially direct influence on its development. The mechanics of continuous media and, later, field theory (thermodynamics, electricity, magnetism, gravitational fields) led to the development of the theory of partial differential equations. The working out of molecular theory, and of statistical physics in general, beginning at the end of the last century, served as an important stimulus for the development of the theory of probability, in particular of the theory of random processes. Through its analytical methods and generalizations, the theory of relativity played a decisive role in the development of Riemannian geometry.

In our time the development of new mathematical theories, such as functional analysis and others, is stimulated by problems in quantum mechanics and quantum electrodynamics, computational problems of technology, statistical questions in physics and technology, and so on. Physics and technology not only pose new problems for mathematics and direct it toward new areas of investigation, but they also provide renewed stimulus for the development of areas of mathematics originally constructed, by and large, from within mathematics, such as Riemannian geometry. Briefly, intensive development of the science requires not only that it proceed to tackle new problems but also that the necessity for their solution be dictated by the needs of the development of society. Many theories have arisen in mathematics in recent times, but only those were developed and received a permanent place in the science which found applications in natural science and technology, or which played the role of important generalizations of those theories which have such applications. Moreover, other theories which found no essential applications; for example, certain refinements of geometrical theories (non-Desarguean and non-Archimedean) have not developed further.

The truth of mathematical results is not, in the end, based on its definitions and axioms, not in the formal rigor of its proofs, but in real applications, i.e., in the final analysis, on practice.

It is necessary to understand, above everything else, that the development of mathematics is the result of the interaction of the logic of the subject matter (reflected in the internal logic of mathematics itself) with the influence of production needs and the links with natural science. This development proceeds in complex ways through the struggle of opposites and includes essential changes in the basic content and form of mathematics. With regard to content, the development of mathematics is determined by its subject matter, but it is impelled basically, and in the final analysis, by the needs of production. Such is the basic law of the development of mathematics.

To be sure, we ought not to forget that this description applies only to the basic laws and that the relation of mathematics to production, generally speaking, is complex. From what we have said above, it would clearly be naive to attempt to base the appearance of any given mathematical theory directly on "production necessities". More than that, mathematics, like every science, possesses a relative independence, its own internal logic, which reflects, as we have emphasized, an objective logic, i.e., a conformity with the laws of the subject matter.

4. Mathematics has always been influenced not only by social production, but by the whole of social conditions in their entirety. Its splendid progress in the epoch of the triumph of classical Greece, the successes of algebra in Italy during the era of the Renaissance, the development of analysis in the period after the English Revolution, the progress of mathematics in France in the period of the French Revolution—all this convincingly demonstrates the continuous connection between mathematical progress and the general progress of society technically, culturally and politically.

This pattern is also clearly exhibited in the development of mathematics in Russia. It is impossible to separate the establishment of an independent Russian school of mathematics, starting with Lobachevsky, Ostrogodsky, and Chebyschev, from the progress of Russian society in its entirety. The time of Lobachevsky is the time of Pushkin and Glinka, the time of the Decembrists, and the blossoming of mathematics was one element of a general progress.

Even more persuasive is the influence of social development in the period after the Great October Socialist Revolution, when investigations of fundamental significance appeared one after another with striking rapidity in many areas: in the theory of sets, topology, number theory, probability theory, the theory of differential equations, functional analysis, algebra, and geometry.

Finally, mathematics has always experienced and still experiences the marked influence of ideology. As with every science, the objective content of mathematics is perceived and interpreted by mathematicians and philosophers in the framework of this or that ideology.

In short, the objective contents of a science are always presented in one ideological form or another; the unity and struggle of this dialectical opposition—objective content and ideological form— play, in the development of mathematics as in every science, a role which is by no means small.

The struggle of materialism, corresponding to the objective contents of science, with idealism, at variance with those contents and distorting their ideas, goes on through the entire history of mathematics. The struggle is clearly marked out in ancient Greece, where the idealism of Pythagoras, Socrates, and Plato is projected against the materialism of Thales, Democritus, and the other philosophers who created Greek mathematics. With the development of a slave-owning order, the upper strata of society separated itself from taking a part in production, considering that to be the destiny of the lower class; and this generated the separation of "pure" science from practice. Only pure theoretical geometry was worthy of the attention of the true philospher. Characteristically, the investigation of certain curves obtained by mechanical means, and even the investigation of conic sections, were considered by Plato to be outside the limits of geometry, since they "do not put us in touch with eternal and incorporeal ideas" but "are used as tools in low and vulgar trades".

A clear example of the struggle of materialism against idealism in mathematics is provided by the activity of Lobachevsky, who advanced and defended a materialist interpretation of mathematics against the idealistic views of the Kantians.

The Russian mathematical school generally is characterized by a materialist tradition. Thus, Chebyschev clearly emphasized the decisive importance of practice, and Lyapunov expressed the approach of the native mathematical school in the following remarkable words: "The more or less general path of theory is the detailed investigation of questions which are of particular importance from the point of view of applications and at the same time present special theoretical difficulties demanding the investigation of new methods and the construction of new scientific principles, and the subsequent generalization of these results and constructions by means of more or less general theory." Generalization and abstraction, not for their own sake but in relation to concrete material; theorems and theories, not for their own sake but in general relation to science, leading in the final analysis to practice—this, indeed, proves to be what is important and rewarding in the whole undertaking.* Such were the aspirations of Gauss and Riemann and other great scholars.

However, with the development of capitalism in Europe, ideological points of view began to work a change in the materialist viewpoint which had reflected the dominant ideology of the expanding bourgeois epoch of the 16th to early 19th centuries. Thus, for example, Cantor (1846-1918), creating the theory of infinite sets, appealed openly to God, declaring in this spirit that infinite sets have absolute existence in the divine intellect. Poincaré, the outstanding French mathematician of the late 19th and early 20th centuries., advanced the idealist notion of "conventionalism", according to which mathematics consists of conventionally agreed-upon schema, taken for convenience as the description of a many-faceted experience. Thus, in the opinion of Poincaré,

* A general understanding of the necessary connection of the different areas of mathematics with each other and with natural science and practice has enormous significance not only for a correct view of mathematics but also for orienting the investigator in the selection of direction and subject of research.

the axioms of Euclidean geometry are no more than agreed-upon conventions, significant because of their clarity, convenience, and simplicity, but which do not conform with reality. For this reason, Poincaré said, physics, for example, would sooner give up the law of rectilinear propagation of light than it would give up Eucliedean geometry. This point of view was refuted by the development of the theory of relativity which, despite all the "simplicity" and "convenience" of Euclidean geometry, led to the result in complete harmony with the materialist ideas of Lobachevsky and Riemann, that the real geometry of space is non-Euclidean.

A variety of tendencies appeared among mathematicians at the beginning of the 20th century as a result of the difficulties arising from the theory of sets and in connection with the necessity for an analysis of the fundamental concepts of mathematics. Agreement was lost as to the way in which the content of mathematics should be understood; different mathematicians came not only to look upon the general foundations of the science in different ways, as had previously been the case, but arrived at different evaluations of the meaning and significance of individual concrete results and arguments. Deductions which were considered meaningful and interesting by one mathematician were declared by another to be devoid of meaning and significance. There arose the idealist currents of "logicalism", "intuitionism", "formalism", etc.

Logicalism asserts that the whole of mathematics is a consequence of the ideas of logic. Intuitionism sees in intuition the source of mathematics and considers only what can be apprehended intuitively to be meaningful. In particular, therefore, it completely denies the significance of Cantor's theory of infinite sets. More than that, intuitionists deny the simple meaning even of such assertions as the theorem that any algebraic equation of nth degree has n roots. For them, this assertion is empty since the method of computing the roots is not indicated. Thus the complete rejection of the objective meaning of mathematics led intuitionists to denigrate as "devoid of meaning" a significant part of mathematics.

The most outstanding mathematician at the beginning of our century, D. Hilbert, undertook to save mathematics from assaults of this type. The essence of his idea was to try to reduce mathematical theories to purely formal operation on symbols according to rules agreed upon previously. The argument was that, in a purely formal approach, all the difficulties would be removed since mathematics would then become the symbols and the rules of acting upon them, without any reference at all to their meaning. This, then, is the aim of formalism in mathematics. According to the intuitionist Brouwer, the truth of mathematics for the formalist exists on paper while for an intuitionist it is in the head of the mathematician.

It is not difficult, however, to see that they are both incorrect, since mathematics, in addition to the fact that it is written on paper and the fact that it is thought by mathematicians, reflects reality, and the truth of mathematics includes within itself the correspondence to objective reality. By divorcing mathematics from material reality, all these tendencies turn out to be idealist.

Hilbert's idea was refuted as a result of its own development. The Austrian mathematician Gödel showed that it is impossible to formalize even arithmetic completely, as Hilbert had believed. Gödel's result clearly revealed the internal dialectic of mathematics, a dialectic which does not permit us to exhaust even one area by formal calculation. Even the simplest infinity, that of the sequence of natural numbers, turned out to be an inexhaustible, finite schema of symbols and their rules of operation. Thus was proved mathematically what Engels had already expressed in a general way when he wrote: "Infinity *is* a contradiction... The removal of the contradiction would be the end of infinity." [*Anti-Duhring,* p. 59.] Hilbert had counted on being able to contain mathematical infinity within the framework of a finite schema, thereby resolving all contradictions and difficulties. This turned out not to be possible.

Under conditions of capitalism, however, conventionalism, intuitionism, formalism, and similar currents are not only preserved but supplemented by new variations of the idealist views of mathematics. Theories related to the logical analysis of the foundations of mathematics are essentially used in several new variants of subjective idealism. Today subjective idealism makes use of mathematics, especially mathematical logic, as well as physics, and for this reason, questions of understanding the foundations of mathematics assume a particular acuteness.

Thus, the difficulties of the development of mathematics under the conditions of capitalism beget an ideological crisis in this science, similar to the crisis in physics, the nature of which was explained by Lenin in his brilliant work, *Materialism and Empirio-Criticism.* The crisis does not at all mean that mathematics in capitalist countries is completely arrested in its development. Many scholars who have assumed clearly idealist positions are responsible for important and at times outstanding successes in the solution of concrete mathematical problems and in the development of new mathematical theories. It suffices to refer to the brilliant development of mathematical logic.

The radical defect of the mathematical views propagated in the capitalist countries lies in their idealism and metaphysics: separating mathematics from reality and neglecting its real development. Logicism, intuitionism, formalism, and other similar currents single out one or another aspect of mathematics—its relationship to logic, its intuitive clarity, its formal rigor, etc.—groundlessly exaggerating and absolutizing its meaning, tearing mathematics away from reality and losing sight of it as a whole behind a deep analysis of a single aspect of mathematics. As a result of such one-sidedness, none of these currents, for all the subtlety and profundity of their particular results, can bring us a true understanding of mathematics. In contrast to the various tendencies and shades of idealism and metaphysics, dialectical materialism considers mathematics in its entirety—and thus, as it actually exists, in all the richness and complexity of its connections and development. And particularly because dialectical materialism strives to understand the connections between science and reality in all of their richness and complexity, all the complexity of the development from simple generalizations of experience to high abstraction and from them to practice, precisely because in its very approach to science it remains in constant correspondence with its objective content and its new discoveries, therefore, and in the last analysis, only because of this, it is the only authentic scientific philosophy leading to the correct understanding of science in general and mathematics in particular. □

Literature

On general problems of mathematics the reader is referred to the following articles in *Bol'shaya Sovetskaya Entsiklopediya* [not yet available in English]:

Kolmogorov, A.N., *"Matematika"*. v. 26.
Aleksandrov, A.D., *"Geometriya"*. v. 10.

APPENDIX

Editorial comment on the AMS and political censorship *within* science

Interest stirred around the mathematical world with the 1964 Moscow publication of a book *Mathematics: Its Content, Methods, Meaning.* Everyone agreed that it was a major contribution to communication with the non-mathematician, the collective triumph of 25 creative Soviet authors and editors, each well known to the world community of mathematicians.

No doubt the shock effect of Sputnik, beeping overhead in 1957, helped to get the book translated and published here in 1963 by the American Mathematical Society with support from the National Science Foundation. The English-language version created more excitement and MIT bought the rights, issuing a handsome three-volume edition that proclaimed on the jacket: "There is no work in English that compares with this major survey of mathematics." A thoughtful foreword by the AMS translation editor praised the expository achievement of the Soviet authors and quoted American mathematicians on its great usefulness not only to lay intellectuals but also to scientists and even other mathematicians.

Few were aware that, by omitting two key sections from Chapter I, the AMS translation had eliminated from this work all discussion of the Marxist philosophy and every mention of the Marxist classics that had provided much of the basis for the expository power of the Soviet authors. Also suppressed, of course, was some trenchant criticism of idealist trends in Western mathematics. A careful reading of the chapter as a whole shows that omission of these two sections was not a mere editorial deletion of redundant material but an outrageous abridgement of the readers' right to know, and to judge independently, the philosophical generalizations and summations that clearly had been planned as an integral part of what the author had to say.

Only a quiet footnote at the end of Chapter I acknowledged that two sections had been omitted "in view of the fact that they discuss in more detail, and in the more general philosophical setting of dialectical materialism, points of view already stated with great clarity in the preceding sections." We suggest that the interested reader personally compare the two sections published here with the seven sections of the AMS version to see exactly how much suppression has been concealed by this seemingly candid footnote.

It is important to place responsibility correctly for such a covert and insidious act of censorship by an important scientific society, with all the attendant and inescapable political implications. No doubt personal responsibility attaches to S.H. Gould, the official AMS translation editor. But events have shown that the leadership of the AMS itself bears the primary responsibility. This became explicit after the matter was brought before the AMS Council at its meeting of 15 August 1977 by Judy Green of Silver Spring, Maryland, then a member of the Council.

Reportedly, at that meeting the Council seemed to agree with Professor Green that the omission constituted political censorship and should be corrected. But action was postponed until the Council meeting of 3 January 1978 at which the Executive Committee recommended, on the basis of a split vote, that the AMS not translate the two omitted sections because of 1) the difficulty of distributing such a translation to the purchasers of the book and 2) the question of "whether in a changed political climate the author would want to have it translated". The AMS Council agreed, though neither of these trivial arguments addressed the central question of political censorship that deprived AMS members and others of the right to decide for themselves on the philosophical questions dealt with in the omitted sections.

The matter was not brought out in the open until Green wrote a letter [AMS *Notices* 25 (4): 240, June 1978] pointing to the responsibility of the AMS which officially handled the translation and took out the copyright. Her letter stressed the importance of correcting an action that reflected the redbaiting atmosphere of the McCarthyite 1950s, as a result of which some AMS members are still unemployed. Green ended by expressing the hope that the Council would reverse itself and publish the omitted sections in the AMS *Bulletin* since she had found that many colleagues would like to read the material in translation.

We hope that word gets around on the availability of these two essays in *Science and Nature* though we think it would have been far preferable that the AMS had demonstrated its integrity by doing the translation and publishing. And we hope that the AMS members will not let the censorship issue be forgotten. The following questions might be addressed quite forcefully to the AMS leadership: Why was an act of censorship upheld that violates every tradition of free scientific inquiry? Why was the mathematical community not permitted to judge for itself the merits of the Marxist philosophical ideas expressed in these two essays? Does not professional *self-censorship* of this kind contribute objectively to the current rightist efforts toward reviving McCarthyite war hysteria? □

Grandmother resolves a contradiction ------------------

On this theme of division there is a humorous question which is extraordinarily instructive. Grand mother has bought three potatoes and must divide them equally between two grandsons. How is she to do it? The answer is: make mashed potatoes.

The joke reveals the very essence of the matter. Separate objects are indivisible in the sense that, when divided, the object almost always ceases to be what it was before, as is clear from the example of "thirds of a man" or "thirds of an arrow." On the other hand, continuous and homogeneous magnitudes or objects may easily be divided and put together again without losing their essential character. Mashed potatoes offers an excellent example of a homogeneous object, which in itself is not separeted into parts but may nevertheless be divided in practice into as small parts as desired. Lengths, areas, and volumes have the same property. Although they are continuous in their very essence and are not actually divided into parts, nevertheless they offer the possibility of being divided without limit.

Here we encounter two contrasting kinds of objects: on the one hand, the indivisible, separate, discrete objects; and on the other, the objects which are completely divisible, and yet, are not divided into parts but are continuous. Of course, these contrasting characteristics are always united, since there are no absolutely indivisible and no completely continuous objects. Yet these aspects of the objects have an actual existence, and it often happens that one aspect is decisive in one case and the other in another.

In abstracting forms from their content, mathematics by this very act sharply divides these forms into two classes, the discrete and the continuous.

The mathematical model of a separate object is the unit, and the mathematical model of a collection of discrete objects is a sum of units, which is, so to speak, the image of pure discreteness, purified of all other qualities. On the other hand, the fundamental, original mathematical model of continuity is the geometric figure; in the simplest case the straight line.

– Aleksandrov, Kolmogorov and Lavrent'ev, *Mathematics: Its Content, Methods, and Meaning.* MIT Press 1969, p. 32.